Mass, Impetus and Force by Symplectic Realizations of the Static Group

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Abstract

We show by symplectic realizations of the one dimensional Static group G that the maximal G-elementary system is a a massive particle under an invariant force f participating in the linear momentum and an invariant impetus I participating in the change of position. Moreover the system is characterized by four physical quantities : a mass m ,a boost $u = \frac{I}{m}$, an acceleration $a = \frac{f}{m}$ and an internal energy U. The hamiltonian of the system is a sum of the potential energy $V = -fq$ and a contribution pu of the impulse.

1 Introduction

Following [1], there are eleven physically plane geometries governed by the eleven Lie algebras. Following [2] and [1] these algebras are generated by K for the boosts, P for space translations and E for time translations whose physical dimensions are respectively that of an inverse of a velocity , an inverse of a length and an inverse of a duration . We distinguish within these Lie algebras

- two simple Lie algebras which are the de Sitter Lie algebras $dS_\pm$ and defined by the Lie brackets

$$[K, P] = \frac{E}{c^2} \quad , \quad [K, E] = P \quad , \quad [P, E] = \pm\omega^2 K \tag{1}$$

where c is a constant velocity while ω is a constant frequency.

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- five solvable Lie algebras which are

  the two Newton-Hooke Lie algebras $N_{\pm}$ defined by the Lie brackets

$$[K,P]=0 \quad , \quad [K,E]=P \quad , \quad [P,E]=\pm\omega^2 K \tag{2}$$

  the Poincaré Lie algebra P defined by the Lie brackets

$$[K,P]=\frac{E}{c^2} \quad , \quad [K,E]=P \quad , \quad [P,E]=0 \tag{3}$$

  and

  the two Para-Poincaré Lie algebras $P_{\pm}$ defined by the Lie brackets

$$[K,P]=\frac{E}{c^2} \quad , \quad [K,E]=0 \quad , \quad [P,E]=\pm\omega^2 K \tag{4}$$

- three nilpotent Lie algebras which are

  the Galilei Lie algebra $\mathcal{G}$ defined by the Lie brackets

$$[K,P]=0 \quad , \quad [K,E]=P \quad , \quad [P,E]=0 \tag{5}$$

  the Carroll Lie algebra $\mathcal{C}$ defined by the Lie brackets

$$[K,P]=\frac{E}{c^2} \quad , \quad [K,E]=0 \quad , \quad [P,E]=0 \tag{6}$$

  and

  the Para-Galilei Lie algebras $\mathcal{G}'$ defined by the Lie brackets

$$[K,P]=0 \quad , \quad [K,E]=0 \quad , \quad [P,E]=\omega^2 K \tag{7}$$

- An abelian Lie algebra which is the static Lie algebra St for which all the above Lie brackets vanish .

Note that a general element of a one parameter Lie group generated by X take the form $exp(xX)$.This means that xX must be dimensionless and then that the physical dimension of the parameter x must be be the inverse of that of X. For that reason the parameters v, x and t associated respectively to K, P and E will have the dimensions of a velocity, a length and a duration. Their duals will be a static momentum k, a linear momentum p and an energy e. The relations between this three physical quantities depend on the structure of the Lie algebras and can be determined by the study of the

strongly hamiltonian realizations of the corresponding Lie groups ([6] and references therein). This paper study the Static group case which has never been studied. In two other papers we study the nilpotent and the solvable cases. One will study the Carroll group and the Para-Galilei group and will compare the results with those obtained for the Galilei group that we revisit for completeness. The other will study the Para-Poincaré groups which also have not been studied and compare the results with those obtained for the Newton-Hooke ([9] and references therein)and the Poincaré groups [8] that also we revisit for completeness.

2 The Static group and its central extension

As said in the previous section, the one spatial dimensional Static Lie algebra $\mathcal{G}$ is the abelian one generated by K for the boosts , P for space translations and E for time translations. The general element of the connected Static group G can be written a $exp(vK+xP+tE)$ where the parameters v , x and t are respectively the velocity parameter, the space translations parameter and the time translations parameter. We can represent the element of G by the triplet $g = (v, x, t)$ and the multiplication law for the static group is then

$$(v, x, t)(v', x', t') = (v + v', x + x', t + t') \tag{8}$$

which means that $G = (\mathbf{R}^3, +)$.
We can verify that the central extension $\widehat{\mathcal{G}}$ of the Static Lie algebra is generated by K, P, E , M, F and Y such that

$$[K, P] = M \ , \ [K, E] = Y \ , \ [P, E] = F \tag{9}$$

This means that M and F have $L^{-2}T$ and $L^{-1}T^{-1}$ as physical dimension while the dimension of Y is that of P. Let us write the general element of the connected central extension $\hat{G}$ of the group G as $exp(\xi M + \zeta F + yY)exp(vK + xP + tE)$ and let us represent the general element of $\hat{G}$ by $\hat{g} = (\alpha, g)$ where $\alpha = (\xi, \zeta, y)$. Use of the Baker-Campbell-Hausdorff [4] give rise to that the multiplication law for $\hat{G}$

$$(\alpha, g)(\alpha', g') = (\alpha + \alpha' + c_1(g, g'), gg') \tag{10}$$

where the cocycle c_1 is such that

$$2c_1(g, g') = (vx' - v'x, xt' - t'x, vt' - v't) \tag{11}$$

Definition of $b : G \to \mathbf{R}^3$ by

$$b(g) = \frac{1}{2}(vx, xt, vt) \tag{12}$$

permit us to verify that the cocycle c_2 given by

$$2c_2(g, g') = (vx' + v'x, xt' + t'x, vt' + v't) \tag{13}$$

is trivial [6] and [7]. The cocycle c_1 is then equivalent to $c = c_1 + c_2$ given by

$$c(g, g') = (vx', xt', vt') \tag{14}$$

and the multiplication law (10) is equivalent to

$$(\alpha, g)(\alpha', g') = (\alpha + \alpha' + c(g, g'), gg') \tag{15}$$

which is explicitly

$$\begin{aligned}&(\xi, \zeta, \eta, v, x, t)(\xi', \zeta', \eta', v', x', t')\\ &= (\xi + \xi' + vx', \zeta + \zeta' + xt', y + y' + vt', v + v', x + x', t + t')\end{aligned} \tag{16}$$

3 $G-$elementary Systems

We verify from (16) that the adjoint action of G on the central extension $\widehat{\mathcal{G}}$ of $\mathcal{G}$ is

$$\begin{aligned}&Ad_{(v,x,t)}(\delta\xi, \delta\zeta, \delta y, \delta v, \delta x, \delta t)\\ &\quad = (\delta\xi + v\delta x - x\delta v, \delta\zeta + x\delta t - t\delta x, \delta y + v\delta t - t\delta v, \delta v, \delta x, \delta t)\end{aligned} \tag{17}$$

If the duality is defined by

$$\begin{aligned}&< (m, f, I, k, p, e), (\delta\xi, \delta\zeta, \delta y, \delta v, \delta x, \delta t) >\\ &\qquad = m\delta\xi + f\delta\zeta + I\delta y + k\delta v + p\delta x + e\delta t\end{aligned} \tag{18}$$

and if the right hand side of (18) has action as physical dimension, then m,f, I,k,p and e have respectively the physical dimensions of a mass, a force, an impetus, a static momentum (mass times position) ,a linear momentum and an energy.
We then verify that the coadjoint action of G on the dual $\widehat{\mathcal{G}}^*$ of $\widehat{\mathcal{G}}$

is given by

$$Ad^*_{(v,x,t)}(m,f,I,k,p,e)$$
$$= (m,f,I,k+mx+It,p-mv+ft,e-fx-Iv) \quad (19)$$

Note that the impetus I produces the energy Iv and the static momentum It .It does not produces linear momentum.
We verify from (9) that the Kirillov form is

$$K_{ij}(m,I,f) = \begin{pmatrix} 0 & m & I \\ -m & 0 & f \\ -I & -f & 0 \end{pmatrix} \quad (20)$$

Each orbit being an elementary system, we distinguish eight kind of elementary systems. Four massive ones corresponding to $m \neq 0$ and four massless ones corresponding to $m = 0$. Among the massive ones as well among the massless ones there two accelerated systems and two free systems. Moreover we distinguish among the accelerated as well among the free ones , two boosted systems and two static systems.

4 Physics of the orbits

In this section we study the physics of the eight elementary systems associated to the Static group.

4.1 Massive Systems

These systems correspond to the case $m \neq 0$. Among them we distinguish those which are accelerated ($f \neq 0$) from the free ones ($f = 0$).

4.1.1 Accelerated Boosted Massive Systems "ABS"

This system corresponds to the case $I \neq 0$. It is characterized by four invariants, three of them m,f and I being trivial, the fourth one U being

$$U = e - pu + fq \equiv e - pu + ka \quad (21)$$

where

$$u = \frac{I}{m} \quad ,q = \frac{k}{m} \quad ,a = \frac{f}{m} \quad (22)$$

Note that u is a boost while a is an acceleration.
One also verify that the coadjoint orbit is a symplectic manifold endowed with the symplectic form

$$\sigma = dp \wedge dq \tag{23}$$

The action of the static group on $C^\infty(\mathcal{O}_{(m,\ f,\ I,\ U)}, \mathbf{R})$ is given by

$$(D_{(v,\ x,\ t)}\psi)(p,q) = \psi(p + mv - ft, q - \frac{I}{m}t - x) \tag{24}$$

We see that the force f participate in the linear momentum while the impetus I participate in the change of position. We then verify that the Static Lie algebra is realized by the hamiltonian vector fields

$$D(K) = m\frac{\partial}{\partial p} \quad , \quad D(P) = -\frac{\partial}{\partial q} \quad , \quad D(E) = -f\frac{\partial}{\partial p} - u\frac{\partial}{q} \tag{25}$$

The components of the corresponding momentum are then a static momentum

$$\mu(K) = mq, \tag{26}$$

a linear momentum

$$\mu(P) = p, \tag{27}$$

and an energy

$$\mu(E) = pu - fq \tag{28}$$

We verify also verify from (24) that the motion equations are

$$f = \frac{dp}{dt} \ , \quad I = m\frac{dq}{dt} \tag{29}$$

which are the usual definition of force and impetus by Newton laws. We also see that the hamiltonian function is up an additive constant

$$H(p,q) = pu - fq \tag{30}$$

The relation (30) shows that the hamiltonian is a sum of a kinetic contribution pu of the impetus and a potential energy $-fq$.
The orbit describes then a constantly boosted massive particle under a constant force f.

4.1.2 Accelerated massive static system "ASS"

The difference with the previous case is that $I = 0$ and then that the internal energy is

$$U = e - fq \tag{31}$$

We can then denote the orbit by $\mathcal{O}_{(m,f,U)}$. It is still endowed with (23) and the action of the Static group on $C^\infty(\mathcal{O}_{(m,f,U)}, \mathbf{R})$ is

$$(D_{(v,\ x,\ t)}\psi)(p,q) = \psi(p + mv - ft, q - x) \tag{32}$$

The motion equations are now

$$f = \frac{dp}{dt}\ ,\quad \frac{dq}{dt} = 0 \tag{33}$$

while the hamiltonian is, up an additive constant, the potential energy

$$H(p,q) = -fq \tag{34}$$

From (33) we can say that *the orbit is a static massive particle under a constant force f.*

4.1.3 Boosted Free Massive System "BFS"

This case corresponds to $f = 0$ and $I \neq 0$. The difference with the accelerated massive system with impetus is that $f = 0$. It is then characterized by the mass m, the impetus I and the internal energy

$$U = e - pu \tag{35}$$

We can denote the orbit by $\mathcal{O}_{(m,I,U)}$ .It is endowed with (23) and the action of the static group on it is such that

$$(D_{(v,\ x,\ t)}\psi)(p,q) = \psi(p + mv, q - ut - x) \tag{36}$$

The motion equations are

$$\frac{dp}{dt} = 0\ \ ,\quad I = m\frac{dq}{dt} \tag{37}$$

while the Hamiltonian is up an additive constant

$$H(p,q) = pu \tag{38}$$

which is a kinetic energy.
The orbit describes a constantly boosted free massive particle .

4.1.4 Free Massive Static System "FSS"

In this case the energy e dual to time translations is a trivial invariant and the orbit can be denoted $\mathcal{O}_{(m,e)}$. It is also endowed with (23) and the action of the Static group on it is such that

$$(D_{(v,\ x,\ t)}\psi)(p,q) = \psi(p + mv, q - x) \tag{39}$$

The motion equations are

$$\frac{dp}{dt} = 0 \quad , \quad \frac{dq}{dt} = 0 \tag{40}$$

while the hamiltonian is the invariant one $H = e$. The equation (40) tell us that *the orbit is a free static particle.*

4.2 Massless Systems

These systems correspond to the cases where $m = 0$.As for the massive systems we distinguish among them those which are accelerated ($f \neq 0$) from the non-accelerated ($f = 0$).

4.2.1 Boosted Massless system under a force "BSF"

In this case the system is characterized by the force f, the impetus I and

$$k_0 = k - \frac{p}{\omega} \tag{41}$$

where $\omega = \frac{f}{I}$ is an invariant frequency. We can denote the corresponding orbit by $\mathcal{O}_{(f,I,k_0)}$. It is endowed with the symplectic (23) form with

$$q = -\frac{e}{f} \tag{42}$$

The action of the Static group on $C^\infty(\mathcal{O}_{(f,I,k_0)}, \mathbf{R})$ is

$$(D_{(v,\ x,\ t)}\psi)(p,q) = \psi(p - ft, q - \frac{v}{\omega} - x) \tag{43}$$

while the motion equations are

$$\frac{dq}{dt} = 0 \quad , \quad f = \frac{dp}{dt} \tag{44}$$

We can verify that the hamiltonian is the potential energy

$$H = -fq \tag{45}$$

The orbit describes a boosted massless system under a constant force f.

4.2.2 Massless static system under a force "SSF"

In this case the system is characterize by a invariant force f and k. It is endowed with (23) and the action of the Static group on $C^\infty(\mathcal{O}_{(f,k)}, \mathbf{R})$ is

$$(D_{(v,\ x,\ t)}\psi)(p,q) = \psi(p - ft, q - x) \tag{46}$$

The motion equations are still (44) while the hamiltonian is also (45).
The orbit describes a massless static system under a constant force f.

4.2.3 Boosted Free Massless System "BFS"

In this case the linear momentum p is a trivial invariant and the system is now characterized by the impetus I and p. We denote the corresponding orbit by $\mathcal{O}_{(I,p)}$. It is endowed with the symplectic form

$$\sigma = de \wedge d\tau \tag{47}$$

where

$$\tau = \frac{k}{I} \tag{48}$$

The action of the Static group on $C^\infty(\mathcal{O}_{(I,p)}, \mathbf{R})$ is

$$(D_{(v,\ x,\ t)}\psi)(e,\tau) = \psi(e + Iv, \tau - t) \tag{49}$$

The motion equations are then

$$\frac{de}{dt} = 0 \ \ , \ \ \frac{d\tau}{dt} = 1 \tag{50}$$

hamiltonian is $H = e$ as in the case of a *free boosted massless particle.*

4.2.4 Free Massless static system "FSS"

This is a trivial (0-dimensional) system consisting of a point (k, p, e) which is *a free static massless particle* on which the static group act trivially.

5 Conclusion

Let us summarize the results in one table for massive systems and in another for massless systems.

- Massive systems

system	$(D_{(v,x,t)}\psi)(p,q)$	motion equations	hamiltonian
ABS	$\psi(p+mv-ft,q-ut-x)$	$f=\frac{dp}{dq}\ ,\ I=m\frac{dq}{dt}$	$H=pu-fq$
ASS	$\psi(p+mv-ft,q-x)$	$f=\frac{dp}{dq}\ ,\ \frac{dq}{dt}=0$	$H=-fq$
BFS	$\psi(p+mv,q-ut-x)$	$\frac{dp}{dq}=0\ ,\ I=m\frac{dq}{dt}$	$H=pu$
FSS	$\psi(p+mv,q-x)$	$\frac{dp}{dq}=0\ ,\ \frac{dq}{dt}=0$	$H=e$

- Massless systems

system	$(D_{(v,x,t)}\psi)(p,q)$	motion equations	hamiltonian
BSF	$\psi(p-ft,q-\frac{v}{\omega}-x)$	$f=\frac{dp}{dq}\ ,\ \frac{dq}{dt}=0$	$H=-fq$
SSF	$\psi(p-ft,q-x)$	$f=\frac{dp}{dq}\ ,\ \frac{dq}{dt}=0$	$H=-fq$
BFS	$\psi(e+Iv,\tau-t)$	$\frac{de}{dt}=0\ ,\ \frac{d\tau}{dt}=0$	$H=e$

We do not include the Free Massless Stactic System which is a fixed point. The realizations on the orbits ABS, ASS, BFS and BSF are faithful while they are unfaithful on the others.

References

[1] M.A.Fernandez Sanjuan, *Group contractions and nine Cayley-Klein geometries*, International Journal of Theoretical Physics,**23**$(1), 1-14, 1984$

[2] J.M.Levy-Leblond, *Journal of Mathematical Physics*, **9**, $1605-, 1968$

[3] J.M.Levy-Leblond, *Ann.Inst.Henri.Poincare*, **3**, $1-, 1965$

[4] A.O.Barut, R.Racczka, *Theory of group representations and applications*, Polish Scientific Publishers, Warszawa, 1977.

[5] A.Wawrzynczyk, *Group Representations and Special functions*,Polish Scientific Publishers, Warszawa, 1984.

[6] J.Nzotungicimpaye, *Galilei-Newton law by symplectic realizations*, Lett. Math.Phys. **15**, $101-110, 1988$

[7] J.Nzotungicimpaye, *Jertk by group theoretical methods*, J.Phys. A:Math.Gen. **27**, $4519-4526, 1994$

[8] J.Nzotungicimpaye, *On a Two-Dimensional Symplectic Space-Time* , arXiv math-ph/0305011

[9] Gadella M., and al.,J.Math.Phys. **32**,1182−, 1991